EVALUATING THE SIGNATURES OF THE MEAN MOTION RESONANCES IN THE SOLAR SYSTEM

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Abstract: *The characteristics of the resonant disturbing function for an asteroid perturbed by a planet in circular orbit are discussed. The location of the libration centers and their dependence with the orbital elements of the resonant orbit are analyzed. A proposed numerical method (Gallardo, 2006a) for computing the strengths of the resonances is revised and applied to the region of the main belt of asteroids showing the relevance of several mean motion resonances (MMR) with several planets.*

Keywords: *Resonances, asteroids*

1 Introduction

There is an interesting diversity of studies on asteroids in mean motion resonances (MMR) with Jupiter and on transneptunian objects in MMR with Neptune. However, not all possible resonances were analyzed neither all the perturbing planets were considered.

The Solar System is in fact covered by a innumerable quantity of possible resonances. If we do not have a method that adequately weighs the strength of each resonance it is laborious to identify which one of the hundreds of possible MMRs that theoretically exist near the semimajor axis of the orbit we are studying is the one affecting the body's motion.

We know by basic celestial mechanics (see also section 2) that the strength of a resonance is approximately proportional to the the mass of the planet and to the eccentricity of the resonant orbit elevated to the order of the resonance, whenever the eccentricity is not very high. Consequently, in general we are not motivated to consider high order resonances neither low mass perturbing planets.

For zero inclination orbits and not very high eccentricities it is possible to compute the widths (somehow related to the strengths) in semimajor axis of the MMRs with the planets as a function of the eccentricity (Dermott and Murray, 1983; Morbidelli *et al.*, 1995; Nesvorný *et al.*, 2002) but no simple theoretical method exists to compute the widths in the case of very eccentric and non zero inclination orbits.

Recently Gallardo (2006a) presented a numerical method to estimate the strength of an arbitrary mean motion resonant orbit with arbitrary orbital elements assuming a circular orbit for the perturbing planet. Based on this principle it is possible to compute the strength of the resonances with all the planets form Mercury to Neptune for all ranges of semimajor axis, from the Sun up to the limits of the Solar System. That tool allowed the author to identify candidates to be in exotic resonances like 6:5 and 1:2 with Venus and 1:2 and 2:5 with Earth. Very recently a numerous population of asteroids in the resonance 1:2 with Mars was also identified (Gallardo, 2007).

We resume here the principles of the method, we analyze some consequences and we apply it to the main belt of asteroids putting in evidence the signatures of some MMRs.

2 The Disturbing Function

Given a planet of mass m_P and radius vector $\mathbf{r}_P$ in an heliocentric frame and a small body at $\mathbf{r}$ the equation of motion is given by

$$\ddot{\mathbf{r}} + k^2 M_\odot \frac{\mathbf{r}}{r^3} = \nabla \mathbb{R} \tag{1}$$

where $\mathbb{R}$ is the disturbing function:

$$\mathbb{R} = k^2 m_P \Big(\frac{1}{\mid \mathbf{r}_P - \mathbf{r} \mid} - \frac{\mathbf{r} \cdot \mathbf{r}_P}{r_P^3} \Big) \tag{2}$$

In order to construct an analytical theory for the dynamics of a small body we need an expression for $\mathbb{R}$. Since Laplace's times classical expressions were constructed as series expansions around $e = 0$ and $i = 0^o$ being one of the most recent versions the one of Ellis and Murray (2000). These expansions have convergence problems for high e (Ferraz-Mello, 1994) and it is necessary to take into account several terms for properly account for high inclination orbits. Planar orbits with high eccentricities are best handled by other expansions like Beaugé's (Beaugé, 1996). For the general problem a local expansion can be constructed in order to study the motion around a small region of the phase space (Ferraz-Mello and Sato 1989; Roig *et al.*, 1998). These *asymmetric* expansions are valid around a small region near the center of the expansion but they can be applied to very high eccentricity orbits allowing the calculation of the precise positions of the libration centers and the periods of the small amplitude librations.

Considering a system composed by the Sun, the planet and a small body with orbital elements $(a, e, i, \varpi, \Omega)$ the classical expression of the expansion for $\mathbb{R}$ is a series of terms of the form:

$$\mathbb{R} = \sum C \cos(\varphi) \tag{3}$$

being C a function of the form

$$C = A(\alpha) e_P^{k1} e^{k2} s_P^{k3} s^{k4} \tag{4}$$

with $s = \sin(i/2)$, $A(\alpha)$ being a function of $\alpha = (a/a_P)^{\pm 1}$ with positive exponent for resonances interior to planet's orbit and negative for exterior ones. The angle φ is defined as

$$\varphi = j_1\lambda_P + j_2\lambda + j_3\varpi_P + j_4\varpi + j_5\Omega_P + j_6\Omega \tag{5}$$

where subscript P denotes the planet. The j_i are integers verifying $\sum j_i = 0$ with $j_5 + j_6$ being always even (D'Alembert rules) and $k_i \geq\mid j_i \mid$. If φ is a quick varying angle the total effect of the corresponding term of $\mathbb{R}$ will vanish in the long term evolution. Also if φ is a slow varying angle but C is vanishingly small the term again will not have a dynamical effect in the motion of the particle. Then if we are interested in a correct description of the long term dynamical evolution of the particle we must take into account all the slow varying terms with non negligible coefficients C; these are the resonant and secular terms. As e_P, e, s_P, s are less than 1 we say that the corresponding term is of order $(k_3 + k_4 + k_5 + k_6)$. It is possible to simplify the analysis taking a circular orbit for the planet with zero inclination, that means $e_P = s_P = 0$. With this approximation the number of terms involved in (3) drop considerably. It is also possible to take into account in $\mathbb{R}$ the terms due to all planets.

Under these hypothesis and taking into account D'Alembert rules a q-order resonance $|p + q| : |p|$ with p and q integers occurs when the general critical angle

$$\sigma_j = (p + q)\lambda_P - p\lambda - (q - 2j)\varpi - 2j\Omega = \sigma + 2j\omega \tag{6}$$

librates or have a slow time evolution, where

$$\sigma = (p + q)\lambda_P - p\lambda - q\varpi \tag{7}$$

is the principal critical angle and j is an integer positive or negative. Due to the very slow time evolution of the angles (ϖ, Ω) the librations of σ_j occur approximately for

$$\frac{n}{n_P} \simeq \frac{p+q}{p} \tag{8}$$

where the n's are the mean motions. Then, the formula

$$\frac{a}{a_P} \simeq (1 + m_P)^{-1/3} \Big(\frac{p}{p+q}\Big)^{2/3} \tag{9}$$

defines the location of the resonances with planet P in semimajor axis. At very low eccentricities the time variation of ϖ cannot be ignored and the location of the resonances are shifted (the *law of structure*) respect to Eq. (9). The integer p is the degree of the resonance with $p < 0$ for exterior resonances and $p > 0$ for interior resonances. With this notation the trojans (or co-orbitals) correspond to $p = -1$ and $q = 0$. The resonant motion is generated when there is a strong dependence of $\mathbb{R}$ on σ which must be librating or in slow time-evolution. In this case $\mathbb{R}(\sigma)$ dominate the time evolution of the small body's orbit.

3 Properties of the Resonant Disturbing Function

The limitations imposed by the problems of convergence of the analytical developments motivate the authors to explore the disturbing function numerically. In analogy with Schubart (1968), in order to explore numerically the function $\mathbb{R}$ for a specific resonance defined by a semimajor axis given by Eq. (9) we can eliminate all short period terms on $\mathbb{R}$ computing the mean disturbing function

$$R(\sigma) = \frac{1}{2\pi|p|}\int_0^{2\pi|p|} \mathbb{R}(\lambda_P, \lambda(\lambda_P, \sigma))d\lambda_P \tag{10}$$

for a given set of fixed values of $(e, i, \varpi, \Omega, \sigma)$ where we have expressed $\lambda = \lambda(\lambda_P, \sigma)$ from Eq. (7) with σ as a fixed parameter and where $\mathbb{R}(\lambda_P, \lambda)$ is evaluated numerically from Eq. (2) where $\mathbf{r}_P$ and $\mathbf{r}$ were expressed as functions of the orbital elements and mean longitudes λ_P and λ. We repeat for a series of values of σ between $(0°, 360°)$ obtaining a numerical representation of the resonant disturbing function $R(\sigma)$.

The equations of the resonant motion show that the time evolution of the semimajor axis, da/dt, is proportional to $\partial R/\partial\sigma$ then the shape of $R(\sigma)$ is crucial because it defines the location of stable and unstable equilibrium points. For specific values of (e, i, ω) the minima of $R(\sigma)$ define the stable equilibrium points also known as *libration centers* around which there exist the *librations*, that means oscillations of the critical angle σ. The unstable equilibrium points are defined by the maxima.

At low (e, i) the function $R(\sigma)$ calculated from Eq. (10) is very close to a sinusoid with amplitude proportional to e^q as one can expect from the classical series expansions. At higher eccentricities the orbit approaches to the planet's orbit, $R(\sigma)$ start to depart from the sinusoid and classical series expansions start to fail with some exceptions like Beaugé's expansion. For eccentricities greater than the *collision eccentricity* e_c:

$$e_c \simeq \left|1 - \left(\frac{p+q}{p}\right)^{2/3}\right| \tag{11}$$

the orbit can intersect the planet's orbit and for low inclination orbits two peaks start to appear around the point where $R(\sigma)$ has its maximum for $e < e_c$. If the two peaks can be distinguished then a stable equilibrium point appears between them. For high inclination orbits the intersection between orbits is less probable and soft maxima can appear instead of the peaks. For small inclination orbits the libration centers are almost independent of (i, ω) and can be classified in only three different classes as showed at Table 1.

Resonances of the type 1:n including trojans (that means 1:1) exhibit a similar general behavior. For $e < e_a$, where e_a is certain value verifying $e_a < e_c$, there is a libration center at $\sigma = 180°$ and for $e > e_a$ there appear the *asymmetric* libration centers (Beaugé, 1994) with locations depending not only on e but also on (i, ω). For trojans we have $e_a = e_c = 0$ so the low eccentricity librations around $\sigma = 180°$ do not exist. For $e > e_a$ *horseshoe* (HS) trajectories wrapping the asymmetric librations can exist. These HS trajectories are of the same nature of the horseshoe trajectories in the case of trojans and are only possible for this kind of resonances. In HS trajectories σ is oscillating with high amplitude around 180°. For $e > e_c$ at low inclinations they appear two peaks (unstable equilibrium points) and a stable libration center at $\sigma = 0°$. This last equilibrium point is associated with the known *quasi-satellites* (QS) of the 1:1 resonances (Wiegert *et al.*, 2000).

For low inclination orbits all odd order interior resonances show librations around $\sigma = 0°$, and for $e > e_c$ it appears another libration point at $\sigma = 180°$. Conversely, all interior resonances of order even and all exterior resonances except resonances of type 1:n show librations around $\sigma = 180°$, and for $e > e_c$ it appears another libration point at $\sigma = 0°$.

For high inclination orbits the geometry of the encounters is strongly modified affecting the peaks of $R(\sigma)$ and its shape becomes completely different to the low inclination case becoming the libration centers strongly dependent on ω. Figures 1, 2 and 3 illustrate the dependence of $R(\sigma)$ and its libration centers with (e, i, ω) for the case of the exterior resonance 1:4 with Neptune. In all figures we have taken such units that $k^2 m_{Jup} = 1$.

4 The Strength of the Resonances

For a given resonant orbit defined by parameters (e, i, ϖ, Ω) with a given planet the disturbing function $R(\sigma)$ is determined. Gallardo (2006a) defined the strength function SR as:

$$SR(e, i, \omega) = < R > -R_{min} \qquad (12)$$

being $< R >$ the mean value of $R(\sigma)$ with respect to σ and R_{min} the minimum value of $R(\sigma)$. This definition is in agreement with the coefficients of the resonant terms of the expansion of the disturbing function for low (e, i) orbits because for this case $R(\sigma)$ is a sinusoid with an amplitude given by $< R > -R_{min}$. Then, for low eccentricity and low inclination orbits SR should follow the function e^q. For high (e, i) resonant orbits the strength cannot be calculated by analytical developments and departures form the low eccentricity regime is the rule. Figures 4, 5 and 6 illustrate the dependence of SR with (e, i, ω) for the case of the exterior resonance 1:4 with Neptune.

When $SR \sim 0$ we have $\partial R/\partial\sigma \sim 0$ for all values of σ and then da/dt will not be dominated by resonant terms but by other terms that will generate some time evolution of the semimajor axis and consequently the resonance will be broken, then the resonance will not be dynamically significant or strong. On the contrary, a high value of SR implies a strong dependence of R on σ and the resonant disturbing function $R(\sigma)$ will dominate the motion forcing the semimajor axis to evolve oscillating around the stable equilibrium points or to evolve escaping from the unstable equilibrium points.

Gallardo (2006a) analyzed the shape of $SR(e, i, \omega)$ for several resonances and found that all them can be roughly classified in two classes ($q \leq 1$ and $q \geq 2$) according to the response of SR to the variation of the inclination which is more important for resonances of order 2 or greater. It is possible to understand why the inclination is an important factor for resonances of order 2 or greater. Analytical developments of $R(\sigma)$ in powers of (e, i) show that for a q-order resonance the lowest order resonant terms are of order q in (e, i) (Murray and Dermott, 1999). In particular for trojans (Morais, 1999) and first order resonances the lowest order terms are independent of i. But, for resonances of order $q \geq 2$ the lowest order resonant terms have a dependence with i that make some contribution to $R(\sigma)$ for high inclination orbits (Gallardo, 2006b). In consequence is natural that for resonances of order 2 or greater the resonances are stronger for high inclination orbits because the resonant terms depending on i will show up. On the contrary we cannot expect such behavior for resonances of order 1 or 0 because the resonant terms depending on i have lower relevance.

5 Applications: Identification of the Resonant Signatures

Following the numerical procedure we have described, Gallardo (2006a) presented an atlas of MMRs for the Solar System and identified several objects in unusual resonances with the terrestrial and with the jovian planets. In particular it was found there exist asteroids evolving in the exterior resonances 1:2 and 2:5 with the Earth.

We present here an atlas of resonances for the main belt of asteroids reworked from Gallardo (2007) and we compare it with the distribution of known asteroids (Fig. 7). The atlas was calculated from Eq. (12) considering $e = 0.3$, $i = 10^o$ and $\omega = 60^o$. The histogram of asteroids was elaborated using the osculating orbital elements of ~ 370000 asteroids taken from ASTORB database (ftp://ftp.lowell.edu/pub/elgb/astorb.html) and grouped in bins of 0.001 AU. It is possible to identify the signatures of some well known resonances with Jupiter. In general, a secular evolution inside these resonances drives the eccentricity to values that a collision with the Sun or a close encounter with Mars or Earth remove the asteroid from the resonance so a gap is generated.

The main belt of asteroids is limited at its extremes by the resonances 4:1 (more precisely a secular resonance is responsible for this border) and 2:1 with Jupiter. The depletion effects due to resonances 3:1, 8:3, 5:2, 7:3, 9:4 and also 11:5 with Jupiter are evident. But we can also focus in some signatures not so evident. It is possible to identify some excess of asteroids in the location of certain resonances. In particular, in the histogram there is an excess of around 30% of asteroids at $a \sim 2.419$ AU, exactly where the resonance 1:2 with Mars is located. Note also that the resonance is isolated and consequently not perturbed by others, that means, it should dominate in that region. That population inside the resonance was confirmed via numerical integrations by Gallardo (2007) constituting around a thousand of known asteroids evolving in the resonance. This is the first numerous population that we have knowledge captured in a MMR with a terrestrial planet.

Figure 7 is also showing that resonances 2:5E already studied by Gallardo (2006a) and 3:4M, 3:8E, 2:3M and 1:3E are strong enough to be considered possible reservoirs of asteroids, at least temporarily. This last one probably is strongly perturbed by 4:1J but 2:3M is relatively isolated and 3:8E is at the middle of a considerable population of asteroids, in consequence they should be populated.

As a closing comment, it is evident that analytical theories plus numerical procedures give us at present a quite complete understanding of the MMRs and several features of the distribution of asteroids' populations in the Solar System can be understood in this context.

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Table 1: Stable equilibrium points for low inclination resonant orbits. Resonances of type 1:n except trojans have an equilibrium point at $\sigma = 180^o$ for low eccentricity orbits. For higher eccentricities this point bifurcates in the two asymmetric points. For eccentricities greater than the collision eccentricity, e_c, it appear another new equilibrium point in all resonances. For high inclination orbits this scheme is strongly modified and the argument of the perihelion becomes relevant for the location of the equilibrium points (see Fig. 3).

Resonance Type	σ_0	new σ_0 for $e > e_c$
exterior 1:n	180^o or asymmetric	0^o
exterior others and interior q even	180^o	0^o
interior q odd	0^o	180^o

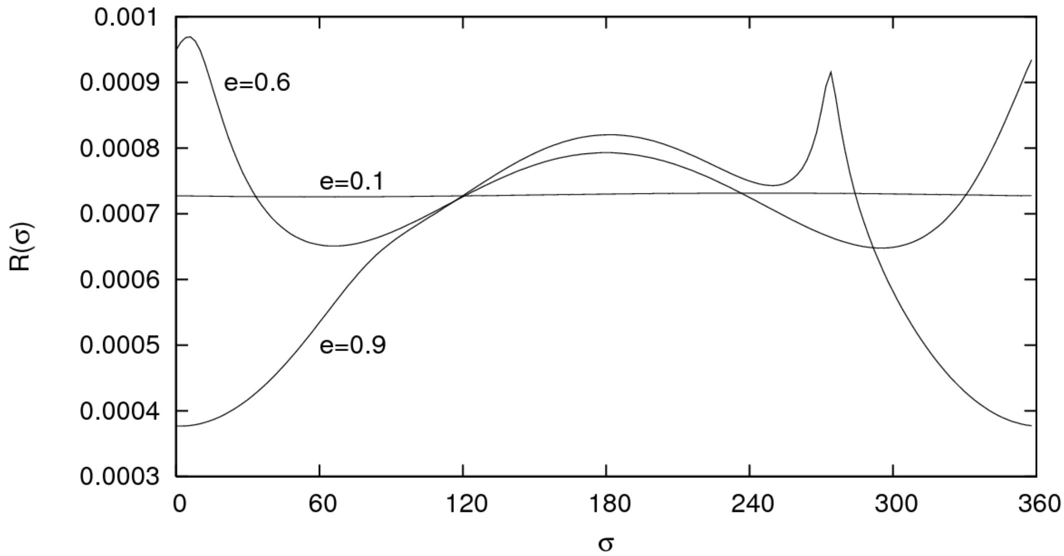

Figure 1: Resonant disturbing function for the resonance 1:4 with Neptune calculated from Eq. (10) for three different values of the eccentricity. In all cases $i = 30^o$ and $\omega = 60^o$. The minima define the location of the stable equilibrium points. Shallow minima at $e = 0.1$ implies low stability.

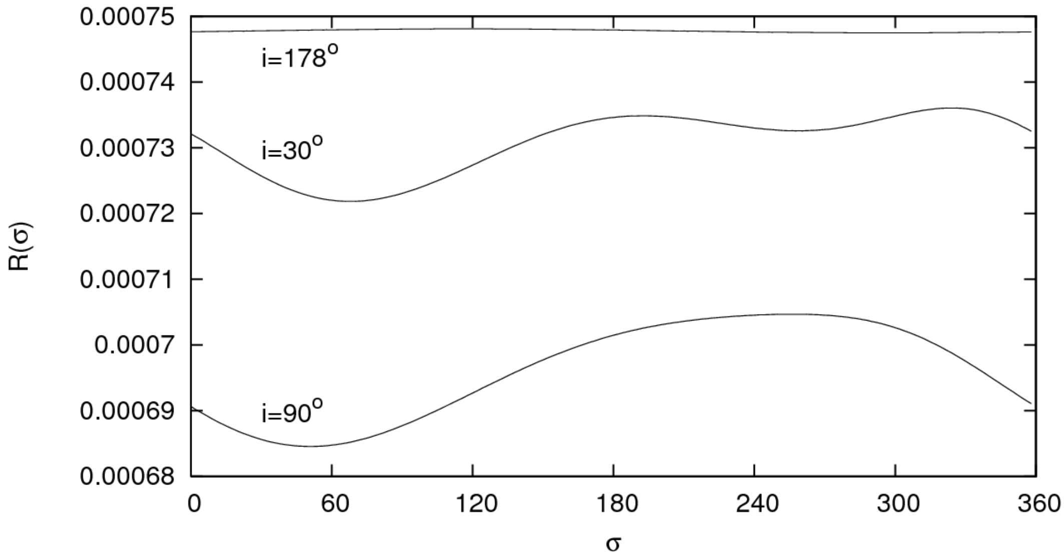

Figure 2: Resonant disturbing function for the resonance 1:4 with Neptune calculated from Eq. (10) for three different values of the inclination. In all cases $e = 0.3$ and $\omega = 60^o$. The minima define the location of the stable equilibrium points. Shallow minima at $i = 178^o$ implies low stability.

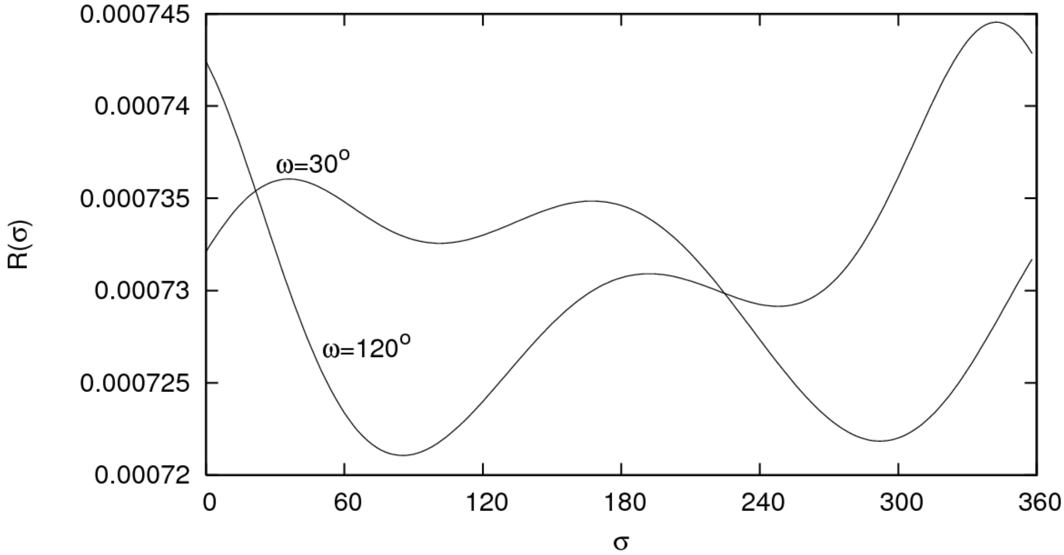

Figure 3: Resonant disturbing function for the resonance 1:4 with Neptune calculated from Eq. (10) for two different values of the argument of the perihelion. In all cases $e = 0.3$ and $i = 30^o$. The minima define the location of the stable equilibrium points.

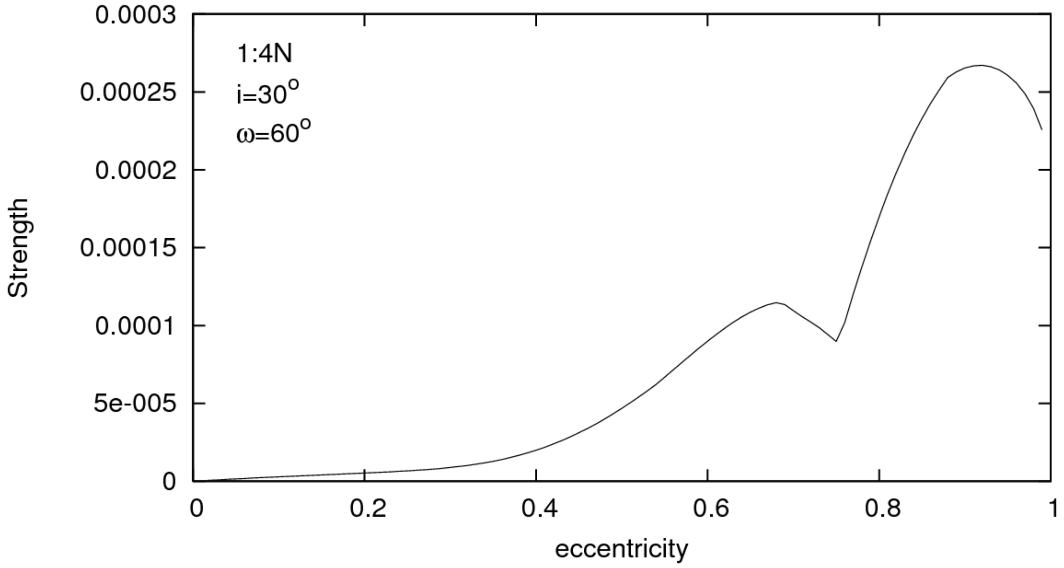

Figure 4: Dependence of the strength $SR(e)$ with the eccentricity for the resonance 1:4 with Neptune calculated from Eq. (12). At low eccentricity regime $SR \propto e^q$ with $q = 3$ in this resonance.

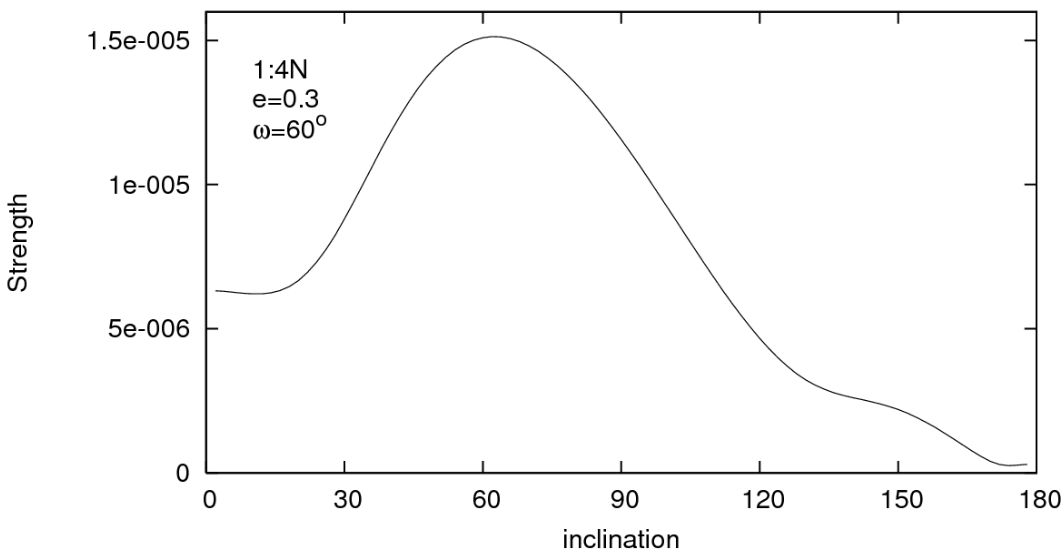

Figure 5: Dependence of the strength $SR(i)$ with the inclination for the resonance 1:4 with Neptune calculated from Eq. (12). The strength is in general greater for high inclination (but direct) orbits.

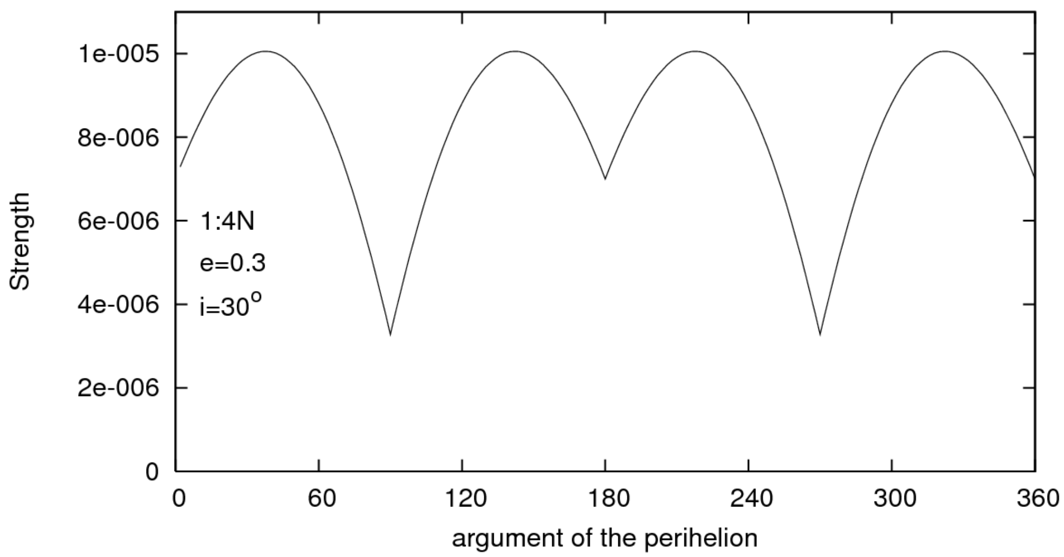

Figure 6: Dependence of the strength $SR(\omega)$ with the argument of the perihelion for the resonance 1:4 with Neptune calculated from Eq. (12). The argument of the perihelion affects the location of the equilibrium points but its influence in the strength is the less important.

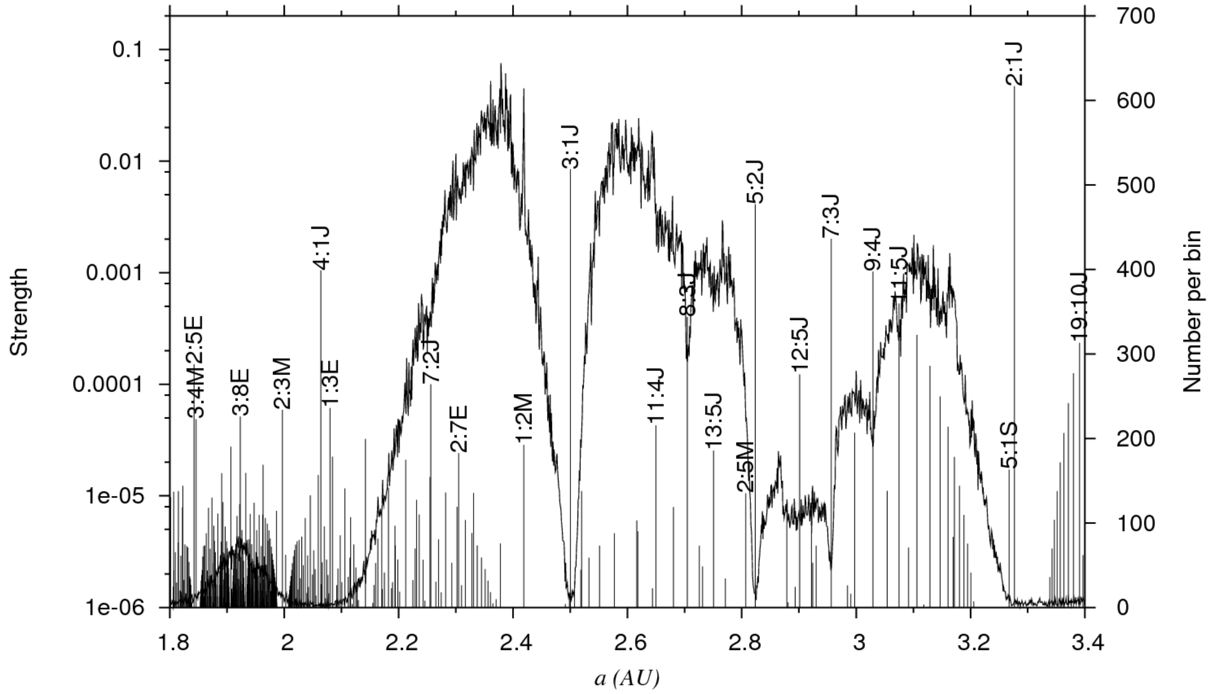

Figure 7: Atlas of the strongest MMRs with all the planets in the region of the main belt of asteroids where strengths were calculated from Eq. (12) assuming $e = 0.3$, $i = 10^o$ and $\omega = 60^o$. Superimposed is showed an histogram of semimajor axes constructed with bins of 0.001 AU. The peak at resonance 1:2 with Mars is clearly distinguished. This figure was recomposed from Gallardo (2007).